Principal Axes for Seakeeping Response Processes


This paper complements an earlier paper by the same author titled “Joint Seakeeping Response Processes for Determining Structural Loads,” presented at the 2002 SNAME Annual Meeting. That paper put forward a variety of methods appropriate to the characterization of joint processes, including phase co-factors, cross co-spectral moments, and joint normal techniques. The present paper exploits cross co-spectral moments to determine principal axes for joint seakeeping processes and explores their behavior and properties in appropriate parameter spaces. The convergence of time domain realizations of these joint processes on the principal axes is also demonstrated through Fourier-Stiljes simulations and conditional probabilities. One example application is to determine the orientation in space of principal axes for vector processes such as translational accelerations in vessel coordinates. As shown by a referenced paper cited in the conclusions, the method can be extended to Von Mises stress and rainflow fatigue.

NOMENCLATURE

Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{\sigma}$</td>
<td>complex valued frequency response function for the $\sigma$-process</td>
</tr>
<tr>
<td>$A_i$</td>
<td>complex valued frequency response function for the $i$th basis process</td>
</tr>
<tr>
<td>$c_i$</td>
<td>linear combination coefficients</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>$i$</td>
<td>imaginary constant, $i = \sqrt{-1}$</td>
</tr>
<tr>
<td>$k$</td>
<td>wave number, $k=\frac{\omega}{g}$</td>
</tr>
<tr>
<td>$m_{yy}^{(n)}$</td>
<td>$n$th statistical moment of the $y$-process</td>
</tr>
<tr>
<td>$p$</td>
<td>probability density</td>
</tr>
<tr>
<td>$P$</td>
<td>cumulative probability</td>
</tr>
<tr>
<td>$S_{yy}$</td>
<td>power spectrum for the $y$-process</td>
</tr>
<tr>
<td>$H_S$</td>
<td>significant wave height</td>
</tr>
<tr>
<td>$T_p$</td>
<td>peak (aka ‘modal’) period</td>
</tr>
<tr>
<td>$T_x$</td>
<td>natural period of the $x$-process</td>
</tr>
<tr>
<td>$T_y$</td>
<td>natural period of the $y$-process</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>generally used to denote a critical process of especial interest; also stress, as in $\sigma_x$ and $\sigma_y$ the components of bi-axial stress</td>
</tr>
<tr>
<td>$\sigma_{yy}^2$</td>
<td>variance of the $y$-process</td>
</tr>
<tr>
<td>$\omega$</td>
<td>circular frequency</td>
</tr>
<tr>
<td>$\theta$</td>
<td>wave direction</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>angle describing orientation of principal axis</td>
</tr>
<tr>
<td>$\psi, \phi, \mu$</td>
<td>nondimensional parameters (see Equations 13-17)</td>
</tr>
</tbody>
</table>

Operators and Notation

The bar over a complex-valued variable denotes the complex conjugate.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x</td>
<td>y)$</td>
</tr>
<tr>
<td>$P(x</td>
<td>y)$</td>
</tr>
<tr>
<td>$\Re$</td>
<td>the real part of a complex quantity</td>
</tr>
<tr>
<td>$\Im$</td>
<td>the imaginary part of a complex quantity</td>
</tr>
</tbody>
</table>
PREFACE

In the brilliant preface to his renowned text, *Wind Waves*, Blair Kinsman distinguishes between *private* science and *public* science. In modern times most published science is public science, using his definition, with results presented in polished, spare, dry, tautological terms. During the Enlightenment, on the other hand, private science prevailed in scientific publication. Authors shared not only their results, but also the thought processes that led to their results, including blind alleys and mistakes.

In this paper I am going to allow myself at least a little private science and disclose that this topic was inspired by a remark a colleague made shortly after my return from the 2002 SNAME Annual Meeting where I had presented a paper titled “Joint Seakeeping Response Processes for Determining Structural Loads.” The ideas presented in this present paper are a natural adjunct to that one and frankly would have been included there if this inspiration had occurred earlier. As an offering to the god of brevity, I will not be repeating much that is contained in that earlier paper – only those portions necessary to the development of principal axes. The two papers are complementary, however, and interested readers will benefit from acquaintance with both.

The colleague whose remark inspired this paper was Ben Ackers. We were both working on a project where total, absolute, translational acceleration was regarded as a critical process. Ben observed that it should be possible to determine principal axes for the translational acceleration process, and I thought …but of course, and I know how to determine those principal axes from the cross co-spectral moments.

The immediate application, and the example given in this technical paper, has been to accelerations, but the point of view and methods described herein should extend to other processes as well. The most obvious extensions would be to motion displacements, velocities, and jerks. It is hoped that yet further applications will occur to some of my readers.

MOTIVATION

Have you ever wondered in what direction the acceleration was greatest as a ship labored in an oblique sea? Or about what axis the rotation was greatest? If so, you were curious about a principal axis for seakeeping processes and answers to your questions may be found herein.

An earlier paper (Hutchison, 2002) observed that the method of cross co-spectral moments is restricted to linear combination processes. The methods described in this paper extend the application of that method to encompass a select set of processes that would normally be regarded as nonlinear and therefore not suitable to it.

CROSS CO-SPECTRAL MOMENTS

The method of cross co-spectral moments was set forth by Mansour (1981) and independently by Hutchison (1982). This method provides a means to directly define the statistics of a process of critical interest that will hereafter be referred to as the $\sigma$-process; provided that the $\sigma$-process is a linear combination of some set of co-joint basis processes.

$$A_\sigma (\omega, \theta) = \sum_{i=1}^{N} c_i A_i (\omega, \theta) \quad (1)$$

The response spectrum for the $\sigma$-process when subject to an incident wave spectrum $S_{\eta\eta} (\omega, \theta)$ is:

$$S_{\sigma\sigma} (\omega, \theta) = A_\sigma (\omega, \theta) \overline{A_\sigma (\omega, \theta)} S_{\eta\eta} (\omega, \theta) \quad (2)$$

where the bar, $\overline{\cdot}$, over a complex quantity denotes the complex conjugate.

The spectral moments of the $\sigma$-process are:

$$m^{(n)}_{\sigma} = \int \int S_{\sigma\sigma} (\omega, \theta) \omega^n \ d\omega \ d\theta = \sum_{i=1}^{N} \sum_{j=1}^{N} \{ c_i c_j m^{(n)}_{ij} \} \quad (3)$$

where: $c_{ij} = c_i c_j$ and

$$m^{(n)}_{ij} = \int \int A_i (\omega, \theta) \overline{A_j (\omega, \theta)} S_{\eta\eta} (\omega, \theta) \omega^n \ d\omega \ d\theta \quad (4)$$

The $m^{(n)}_{ij}$ are the cross spectral moments. In general the cross spectral moments are complex-valued and the off-diagonal elements are complex conjugate pairs (i.e., $m^{(n)}_{ij} = m^{(n)\ast}_{ji}$). The diagonal elements of the cross spectral moments are wholly real. The zero-order diagonal elements are the respective variances of the basis processes (i.e., $m^{(0)}_{kk} = \sigma^2_{kk}$).

As $c_{ij} = c_{ji}$ the imaginary parts of the complex conjugate pairs in (3) cancel. Thus, only the real parts of the cross spectral moments participate in the equation, which are $\Re \{ m^{(n)}_{ij} \}$.

Rayleigh statistics of the $\sigma$-process can be determined in the conventional way from the spectral moments, $m^{(n)}_{\sigma\sigma}$, with the zero-order moment, $m^{(0)}_{\sigma\sigma}$, being the most important. The higher spectral moments may be used to

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1 Jerks are the first time derivative of acceleration, i.e., the time rate of change of acceleration.
determine mean zero-crossing periods and spectral breadth parameters (see, for instance, Hutchison, 2002).

The cross co-spectral moment method permits direct evaluation of the statistics, including extreme values, of any linear combination $\sigma$-process, on equal basis with that accorded the underlying basis processes. The shortcoming of the method is its inability to address $\sigma$-processes that are not linear combinations of the basis processes. It will be shown that by determining principal axes the method of cross co-spectral moments can be extended to a select class of nonlinear processes what will herein be referred to as Pythagorean processes.

PYTHAGOREAN PROCESSES

A short time sample of a joint x-y time series is shown in Figure 1a. A longer sample of the same process is shown in Figure 1b. It is possible to make out detail in Figure 1a that is obscured by the density of the nearly co-located traces in Figure 1b, but the longer sample in Figure 1b provides a better sense of the ultimate distribution of x-y positions.

Our interest is to study the process represented by the distance from the origin to the instantaneous x-y position. The phantom line in Figures 1a and 1b is the theoretical direction along which that distance is expected to be maximum, hereafter discussed as a principal axis.

Pythagorean processes are herein defined to be critical processes that are determined through application of the Pythagorean rule, i.e., the critical process is determined as the square root of the sum of the squares of the underlying basis processes. In 2-D space:

$$\sigma(t) = \sqrt{[x(t)]^2 + [y(t)]^2}$$  \hspace{1cm} (5)

and in 3-D space:

$$\sigma(t) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$$  \hspace{1cm} (6)

Figure 1a: 120 second sample of joint x-y process

Figure 1b: 1200 second sample of joint x-y process
Figure 2: Example 2-D Pythagorean process

Figure 2 illustrates a 2-D Pythagorean process (a different example from that given in Figure 1). The bottom panel of Figure 2 shows time series for X- and Y-processes and the top panel shows the \( \sigma \)-process corresponding to equation 5. Note that the Pythagorean \( \sigma \)-process is positive definite.

Figure 3: Illustrating the determination of sample principal angles for example 2-D Pythagorean process

Figure 3 illustrates the determination of sample principal angles for the example 2-D Pythagorean process of Figure 2. The scale at the left is for the \( \arctan(Y/X) \) process. Superposed without a scale is the Pythagorean \( \sigma \)-process shown also in the top panel of Figure 2. Wherever the \( \sigma \)-process has a local peak the corresponding \( \arctan(Y/X) \) is identified with an \( \circ \) as a sample principal angle.

For the segment of sampled time shown in Figure 3 the sampled principal angles cluster into two groups. One group ranges from -53.5\(^\circ\) (i.e., 306.5\(^\circ\)) to 55.5\(^\circ\) with an average of 10.6\(^\circ\). The second group ranges from 166.7\(^\circ\) to 219.8\(^\circ\) with an average of 193.0\(^\circ\). The difference between the averages of these two sample groups is 182.4\(^\circ\), which is close to the expected 180\(^\circ\).

PRINCIPAL AXES

For each operating and environmental condition there exist principal axes for linear acceleration processes at any field point of interest. In general, these principal axes are not aligned with the body coordinates and the orientation of the principal axes varies with field point location.

The principal axes are a spatial property of the joint stochastic basis processes presumed in the following to be known in orthogonal directions (body axes).

Figure 4 depicts the principal angle, \( \Theta \), for acceleration in the horizontal (x-y) plane.\(^2\) This horizontal plane may be either an earth-fixed-coordinates horizontal plane or a vessel-fixed-coordinates horizontal plane depending on whether the frequency response operators, \( A_x(\omega) \) and \( A_y(\omega) \), are operators for earth-fixed or vessel-fixed accelerations.

Figure 5 depicts the principal angles, \( \Theta_1 \) for azimuth and \( \Theta_2 \) for elevation, for translational acceleration in 3-D space. These angles may be either in earth-fixed coordinates or vessel-fixed coordinates according to the coordinates in which the (complex-valued) frequency

\(^2\) The plane is not restricted to horizontal. Any plane defined by orthogonal basis processes will qualify.
response operators, \( A_x(\omega) \), \( A_y(\omega) \) and \( A_z(\omega) \), are defined.

The key to determining the principal axes is to observe that the linear combination coefficients, \( c_i \), in equations 1 and 3 may initially be treated as unknowns that can be determined in such a way as to maximize the variance of the objective function, \( m_\sigma^{(0)} \). In particular, the \( c_i \) may be trigonometric functions of some initially unknown angles, which, if properly defined, will be the principal axes for the \( \sigma \)-process.

The angles describing the alignment of these principal axes may be determined from the cross co-spectral moments of the body axis acceleration processes. Let the body axis cross spectral moments be defined as follows:

\[
\begin{align*}
    m_{xx} &= \int_0^\infty A_x(\omega) \overline{A_x(\omega)} S_{\eta\eta}(\omega) \, d\omega \\
    m_{yy} &= \int_0^\infty A_y(\omega) \overline{A_y(\omega)} S_{\eta\eta}(\omega) \, d\omega \\
    m_{zz} &= \int_0^\infty A_z(\omega) \overline{A_z(\omega)} S_{\eta\eta}(\omega) \, d\omega \\
    m_{xy} &= \int_0^\infty A_x(\omega) \overline{A_y(\omega)} S_{\eta\eta}(\omega) \, d\omega \\
    m_{xz} &= \int_0^\infty A_x(\omega) \overline{A_z(\omega)} S_{\eta\eta}(\omega) \, d\omega \\
    m_{yz} &= \int_0^\infty A_y(\omega) \overline{A_z(\omega)} S_{\eta\eta}(\omega) \, d\omega
\end{align*}
\]

where: \( S_{\eta\eta}(\omega) \) is the wave power spectral density.

\( A_x(\omega) \), \( A_y(\omega) \) and \( A_z(\omega) \) are the complex-valued frequency response operators for the \( x\)-, \( y\)- and \( z\)-acceleration processes respectively (in vessel coordinates).

A bar (\( \overline{\cdot} \)) over a complex-valued variable indicates the complex conjugate.

The diagonal cross spectral moments \( m_{xx} \), \( m_{yy} \) and \( m_{zz} \) are wholly real and, in general, the off-diagonal moments \( m_{xy} \), \( m_{xz} \) and \( m_{yz} \) are complex. The real parts of \( m_{xy} \), \( m_{xz} \) and \( m_{yz} \) are known as the co-spectral moments and the imaginary parts are known as the quad-spectral moments.

The following nondimensional variables, useful to the development and understanding of principal axes, are defined from these cross spectral moments:

\[
\begin{align*}
    \phi_{xy} &= \arctan \left( \frac{m_{yy}}{m_{xx}} \right) \\
    \phi_{xz} &= \arctan \left( \frac{m_{zz}}{m_{xx}} \right) \\
    \psi_{xy} &= \frac{\Re(m_{xy})}{\sqrt{m_{xx} m_{yy}}} \quad (15a) \quad \mu_{xy} = \frac{\Im(m_{xy})}{\sqrt{m_{xx} m_{yy}}} \quad (15b) \\
    \psi_{xz} &= \frac{\Re(m_{xz})}{\sqrt{m_{xx} m_{zz}}} \quad (16a) \quad \mu_{xz} = \frac{\Im(m_{xz})}{\sqrt{m_{xx} m_{zz}}} \quad (16b) \\
    \psi_{yz} &= \frac{\Re(m_{yz})}{\sqrt{m_{yy} m_{zz}}} \quad (17a) \quad \mu_{yz} = \frac{\Im(m_{yz})}{\sqrt{m_{yy} m_{zz}}} \quad (17b)
\end{align*}
\]

The nondimensional variables \( \phi \) measure the relative dominance of each orthogonal component of the underlying basis response processes. \( \psi \) and \( \mu \) are respectively the nondimensional real and imaginary parts of the cross spectra. Formally, the nondimensional variables \( \psi \) correspond to correlation coefficients (i.e., \( \psi_{xy} \equiv \rho_{xy} \), \( \psi_{xz} \equiv \rho_{xz} \) and \( \psi_{yz} \equiv \rho_{yz} \)).

For 2-D cases the subscripts will be dropped and it will be understood that \( \phi = \phi_{xy} \), \( \psi = \psi_{xy} \) and \( \mu = \mu_{xy} \).

As may be determined by examining Figure 4, the amplitude of a vector along the 2-D principal axis is \( r = x \cos(\Theta) + y \sin(\Theta) \). Consequently \( c_1 = \cos(\Theta) \) and \( c_2 = \sin(\Theta) \). From equation 3 it follows that the principal angle, \( 0^\circ \leq \Theta \leq 180^\circ \), for 2-D acceleration, is that angle that maximizes the following expression:

\[
m_{\sigma\sigma} = \cos^2(\Theta) m_{xx} + 2 \cos(\Theta) \sin(\Theta) \Re(m_{xy}) + \sin^2(\Theta) m_{yy}
\]
As may be observed from Figure 5, the amplitude of a vector along the 3-D principal axis is
\[ r = x \cos(\Theta_1) \cos(\Theta_2) + y \sin(\Theta_1) \cos(\Theta_2) + z \sin(\Theta_2) . \]
Consequently, for the 3-D case \( c_1 = \cos(\Theta_1) \cos(\Theta_2) \), \( c_2 = \sin(\Theta_1) \cos(\Theta_2) \) and \( c_3 = \sin(\Theta_2) \). From equation 3 it follows that the azimuth and elevation of the principal angle for acceleration in 3-D space are those angles, \( 0^\circ \leq \Theta_1 \leq 180^\circ \) and \( 0^\circ \leq \Theta_2 \leq 90^\circ \), that maximize:

\[ m_{xx} = \cos^2(\Theta_1) \cos^2(\Theta_2) m_{xx} + \sin^2(\Theta_1) \cos^2(\Theta_2) m_{yy} \]
\[ + \sin^2(\Theta_2) m_{zz} + 2 \cos(\Theta_1) \sin(\Theta_1) \cos(\Theta_2) \Re(m_{xy}) \]
\[ + 2 \cos(\Theta_1) \sin^2(\Theta_2) \Re(m_{zx}) \]
\[ + 2 \sin(\Theta_1) \cos(\Theta_2) \sin(\Theta_2) \Re(m_{yz}) \]

(19)

General, closed-form, analytical solutions to transcendental equations 18 and 19 are not offered but these equations may be solved to a practical degree of accuracy using systematic solution search methods on a digital computer.

Figure 6 shows the principal angle, \( \Theta \), for 2-D acceleration as a function of \( \phi \) (see equation 13) with parametric dependence on \( \psi \) (see equation 15a).

Providing a different perspective, Figure 7 is a contour plot that depicts contours of constant \( \Theta \) over the of \( \phi-\psi \) plane.

**Figure 6:** Principal angle for 2-D acceleration as a function of \( \phi \) and showing parametric dependence on \( \psi \)
Figure 7: Principal angle for 2-D acceleration

Theta and phi both equal to 45° appears as singular point in Figures 6 and 7. The extreme slope and ‘corner’ that forms in Figure 6 as $\psi \to 0$ is mapped to a smooth process in Figures 7.

**APPLICABILITY**

Principal axes in any number of degrees-of-freedom exist whenever the following conditions are satisfied:

1) The system possesses a Hermitian cross spectral moment matrix.

2) The basis degrees-of-freedom may be regarded as belonging to an orthogonal Euclidian space and are dimensioned such that vector addition is possible and meaningful.\(^3\)

These conditions are satisfied by a very broad and inclusive class of systems, some of which are stationary random process solution sets belonging to all linear marine dynamic systems, as might be represented by coupled or un-coupled sets of second-order ordinary differential equations. Certainly all systems that possess linear frequency response operators (i.e., response amplitude ratios and phase) meet these requirements. Customary practice would extend this set to include weakly nonlinear systems.

It is not uncommon to collect time-domain response data for joint processes, either in a model basin or from full-scale instrumentation. Cross-spectral moment matrices can always be determined from any joint time series. Even when the underlying relationship between response and stimulus (e.g., incident waves) is nonlinear, perhaps even strongly nonlinear, it may be that the frequency dependent phase relationships between the basis responses are sufficiently invariant that principal axes may be determined from the cross spectral moment matrices. The coherency function may be used to obtain insight into the reasonableness of determining principal axes from a given joint data set.

**TYPICAL RESPONSE PROCESSES**

In order to gain an appreciation for the range of typical behaviors of 2-D principal axes, pairs of generic second-order response processes representing dynamic marine systems were developed. The response functions were solutions to the following un-coupled, linear, second-order ordinary differential equations with constant coefficients.

These equations were selected for their simplicity and because they manifest the principal features necessary to represent marine dynamic systems, specifically damped resonant peaks leading to narrow band responses. While the equations are generic and not intended to represent any specific real response function, they could be regarded, for example, as pitch and roll, which are uncoupled (for vessels with a longitudinal plane of symmetry) within the context of linear ship motion theory.

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\(^3\) To satisfy this condition it may be necessary to scale or otherwise transform the native basis processes.
\[ a_x \ddot{x} + b_x \dot{x} + c_x x = f_x(\omega)\zeta_0 e^{i\omega t} \quad (20) \]
\[ a_y \ddot{y} + b_y \dot{y} + c_y y = f_y(\omega)\zeta_0 e^{i\omega t} \quad (21) \]

where: \( \zeta_0 \) the wave amplitude
\( a_x, a_y \) virtual masses
\( b_x, b_y \) damping coefficients
\( c_x, c_y \) restoring coefficients
\( f_x(\omega), f_y(\omega) \) wave forcing functions

The natural periods are given by:
\[ T_x = 2\pi \sqrt{\frac{a_x}{c_x}} \quad (22a) \]
and
\[ T_y = 2\pi \sqrt{\frac{a_y}{c_y}} \quad (22b) \]

And the damping is given by:
\[ b_x = \nu_{CR,x} \sqrt{4a_x c_x} \quad (23a) \]
and
\[ b_y = \nu_{CR,y} \sqrt{4a_y c_y} \quad (23b) \]

where: \( \nu_{CR} \) is the ratio of the damping to critical damping.

In the spirit of Schmitke (1978) the wave forcing functions are given by:
\[ f_x(\omega) = \{c_x - \frac{1}{2}a_x\omega^2 + i b_x \omega\} \frac{\sin\left(\frac{1}{2}kL_x\right)}{\frac{1}{2}kL_x} e^{-kd_x} \quad (24a) \]
and
\[ f_y(\omega) = \{c_y - \frac{1}{2}a_y\omega^2 + i b_y \omega\} \frac{\sin\left(\frac{1}{2}kL_y\right)}{\frac{1}{2}kL_y} e^{-kd_y} \quad (24b) \]

where: \( k = \frac{\omega}{g} \) is the wave number
\( g \) is the acceleration of gravity

and the damping ratios, characteristic lengths (L_x and L_y) and drafts (d_x and d_y) were set as constants as shown in the following table.

<table>
<thead>
<tr>
<th>x-Response Process</th>
<th>y-Response Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_{CR} )</td>
<td>0.05</td>
</tr>
<tr>
<td>L</td>
<td>400 feet</td>
</tr>
<tr>
<td>d</td>
<td>20 feet</td>
</tr>
</tbody>
</table>

Plots of example pairs of response functions (response amplitude operator as a function of frequency) are provided in Figure 8(a-d).

Natural periods for the X-process were varied from 3 to 27 seconds. The response amplitude operators for the X-process are shown in Figure 9a. The natural frequencies for the Y-process were independently varied from 4 to 26 seconds. The response amplitude operators for the Y-process are shown in Figure 9b.

The response functions used in the analyses reported herein were complex valued,\(^5\) but, in order to simplify the presentation, the phase is not shown in Figure 8(a-d) or Figure 9(a-b).

\(^4\) Note that only one-half of the virtual mass is included in the wave forcing functions. This reflects an assumed division of the virtual mass into equal parts physical and hydrodynamic added mass.

\(^5\) Real and imaginary parts, or, alternatively, amplitude and phase.
Figure 8: Response amplitude operators for $T_x=11$ sec and various $T_y$
SAMPLES OBTAINED FROM SIMULATION

Cross co-spectral moments and predicted 2-D principal angle were obtained for each pair of X- and Y-processes. The associated Pythagorean process was simulated using Fourier-Stieltjes integrals and sampled in the time domain.

The Fourier-Stieltjes integrals\(^6\) are given in equations 25 and 26:

\[
X(t) = 9 \int_{0}^{\infty} A_x(\omega) e^{j[\varepsilon(\omega) - \omega t]} \sqrt{2 S_{zz}(\omega)} \, d\omega
\]

\[
Y(t) = 9 \int_{0}^{\infty} A_y(\omega) e^{j[\varepsilon(\omega) - \omega t]} \sqrt{2 S_{yy}(\omega)} \, d\omega
\]

where:

\[
A_x(\omega) = \frac{x(\omega)}{\zeta_0}
\]

is the complex-valued response amplitude operator for the X-process, given by the solution to equation 20

\[
A_y(\omega) = \frac{y(\omega)}{\zeta_0}
\]

is the complex-valued response amplitude operator for the Y-process, given by the solution to equation 21

\[
\varepsilon(\omega)
\]

is a random phase uniformly distributed:

\[
0 \leq \varepsilon < 2\pi
\]

\[
S_{zz}(\omega)
\]

is the wave spectral density function

In the present studies the Bretschneider two-parameter formulation for the wave spectral density function was used:

\[
S_{zz}(\omega) = A\omega^{-5} \exp\{-B\omega^{-4}\}
\]

where:

\[
A = 487.0626 \frac{H_s^2}{T_p^4}
\]

\[
B = \frac{1948.2444}{T_p^4}
\]

\[
H_s
\]

is the significant wave height

\[
T_p
\]

is the peak (aka ‘modal’) period

The present studies explore properties that are insensitive to the significant wave height, so \(H_s = 1.0\) was applied in all cases.
SAMPLE CORRELOGRAMS

Figure 10(a-h) shows sample correlograms between X and Y sampled at the peak of the associated Pythagorean process. Figure 10 shows a single case determined by natural periods \( T_x=8 \) and \( T_y=11 \). The differences in the correlograms are all a consequence of the variation in the peak period of the wave spectra, \( T_P \).

The predicted principal angle is shown in each case by a phantom line, and the value of the principal angle is provided in the lower right-hand corner of each correlogram. Also provided in each correlogram is the sample correlation coefficient, \( \rho \).

When the absolute value of the correlation coefficient is high (approaches unity), as for example in 10a, the scatter of the points about the predicted principal angle is very tight, and the radius to the sample points is clearly greatest along the principal axis.

When the absolute value of the correlation coefficient is low, as it is in 10d, the scatter of the points about the predicted principal angle is large, and it is less clear that the radius to the sample points is greatest along the principal axis.
PREDICTED PRINCIPAL ANGLES

Figure 11 shows the predicted principal angle as a function of peak period, \( T_P \), for the case where \( T_x=8 \) and \( T_y=11 \). Also shown in Figure 11 are the sampled principal angles for the cases corresponding to Figure 10(a-h). The sampled principal angles fall on the predicted line except for cases where \( T_P < 5.7 \) seconds. However, this is only a result of the 180° ambiguity inherent in the modulo arithmetic for the orientation of the principal axis. Adding or subtracting any integer multiples of 180° does not alter the orientation of the principal axis described. By adding 180° to the sampled segment with \( T_P < 5.7 \) seconds, a continuous function without ordinate or slope discontinuity is obtained.

Figure 12 shows a complete family of predicted principal angles for the case where \( T_x=8 \) seconds. The parametric family shown is for \( T_y=3, 5, \ldots 21 \) seconds. The behavior of the surface is quite complex for \( T_P < 8 \) seconds and \( T_y < 7 \) seconds. Judging from the curves shown, a better behaved surface is obtained at all \( T_P \) for \( T_y > 7 \) seconds, and above \( T_P = 8 \) seconds for \( T_y < 7 \) seconds. There is a clear asymptotic tendency with increasing \( T_P \). Similar surfaces are obtained for other values of \( T_x \).
Figure 11: Principal angle for $T_x=8$ sec, $T_y=11$ sec as a function of peak period, $T_p$, for the Bretschneider spectrum.
Figure 12: Parametric dependence on $T_y$ of principal angle for $T_x=8$ sec, as a function of peak period, $T_p$, for the Bretschneider spectrum
CORRELATION

As previously noted, the nondimensional parameter $\psi$ (psi) formally corresponds to the correlation coefficient between the X- and Y-processes. Figure 13 compares the value of $\psi$ estimated from the cross co-spectral moments, using equation 15a, to the sample correlation coefficient obtained from the time domain simulated X- and Y-processes. Examples are shown in Figure 10(a-h). In the correlogram shown in Figure 13, all correspond to the case where $T_x=8$ and $T_y=11$. The slope of the linear regression fit between time domain sampled correlation and the value of $\psi$ predicted from the cross co-spectral moments is 1.004, while the scatter of the sample points about this regression line as measured by $r^2$ is 0.984. A perfect correlation would be a slope of 1.0, a y-intercept of zero, and a sample correlation of 1.0. These sampled measures lend support to the contention that $\psi$ is indeed the population correlation coefficient between X- and Y-processes when sampled at the peaks of the associated Pythagorean process.

Figure 14 is similar to Figure 13 and shows a correlation between the value of $\psi$ estimated from the cross co-spectral moments and the sample correlation coefficient obtained from the time domain simulated X- and Y-processes. However, Figure 14 shows the correlation of 3432 points belonging to thirteen different values of $T_x$, twelve different values of $T_y$, and twenty-two values of $T_P$. Again the correlation is excellent, with $r^2=0.969$ and a slope of 1.008.

Figure 15 shows a complete family of correlation parameters, $\psi$, for the case where $T_x=8$ seconds. The parametric family shown is for $T_y=5, 7, \ldots, 19$ seconds. The behavior of the surface is complex for $6 < T_P < 14$ seconds and $T_y>11$ seconds. Similar surfaces are obtained for other values of $T_x$.

![Correlogram comparing sample correlation coefficient, $\rho_{xy}$, and $\psi$ for $T_y$ of principal angle for case where $T_x=8$ sec, $T_y=11$ sec (variation in $\psi$ a consequence of different $T_P$)](image-url)

**Figure 13:** Correlogram comparing sample correlation coefficient, $\rho_{xy}$, and $\psi$ for $T_y$ of principal angle for case where $T_x=8$ sec, $T_y=11$ sec (variation in $\psi$ a consequence of different $T_P$)
**Figure 14:** Correlogram comparing sample correlation coefficient, $\rho_{xy}$, and $\psi$ for all $(T_x, T_y, T_P)$ cases

**Figure 15:** Parametric dependence on $T_y$ of correlation parameter, $\psi$, for $T_x=8$ sec, as a function of peak period, $T_P$, for the Bretschneider spectrum
CONVERGENCE

Principal axes for joint seakeeping processes provide another example of an important property associated with joint Gaussian processes in general: the property of convergence. Convergence refers to the tendency towards decreased deviation of realizations from a predicted joint property as the realizations become more extreme. This is a valuable property that increases and supports the value of the predicted joint properties in our analyses.

Figure 16 shows 1257 sampled peaks of the Pythagorean principal angle process where $T_x=8$ seconds, $T_y=11$ seconds and $T_p=6$ seconds. The abscissa is a nondimensional measure of the radius to the sampled points:

$$\tau(t) = \frac{\sqrt{x(t)^2 + y(t)^2}}{\sqrt{m_{xx} + m_{yy}}} \quad (28)$$

Equation 28 is a nondimensional form of the Pythagorean process originally given in equation 5. The values of $\tau$ (tau) in Figure 16 correspond to those instants in time where $\sqrt{x(t)^2 + y(t)^2}$ is locally maximum (a peak), as illustrated and discussed in Figure 3.

The parameters of the case shown in Figure 16 are: $\phi=1.96^\circ$ and $\psi=-0.042$, resulting in a principal angle of $\Theta=-0.453^\circ (=179.547^\circ)$. The ordinates of the points shown in Figure 16 are the absolute value of the deviations of each sampled principal angle from the predicted principal angle:

$$\delta(t) = |\tan^{-1} \frac{y(t)}{x(t)} - \Theta| \quad (29)$$

Also shown in Figure 16 is a 20 point moving average over the sample set and a power equation trendline. The power equation appears as smooth, lower bound approximation to the moving average. Both trendlines have high values at low values of $\tau$ and decrease with increasing $\tau$. At low values of $\sigma$ the deviations are capped by an upper bound limit of 90°. The convergence property of interest is the tendency for the deviation from the predicted principal axis to diminish with increasing $\tau$.

**Figure 16:** Scatter diagram of sampled peaks of the Pythagorean process for the case where $T_x=8$ sec, $T_y=11$ sec and $T_p=6$ sec, showing convergence through moving average and power equation trendlines.
Figure 17: Example power equation trendlines showing convergence for various $T_P$, where $T_x=8$ sec, $T_y=11$ sec

Figure 17 shows the power equation trendlines for sixteen different $T_P$ values for the case where $T_x=8$ seconds and $T_y=11$ seconds. Also provided for selected curves are the corresponding values of $\psi$. Considerable variability may be observed in the ordinal location of the trendlines, but all cases clearly display the convergence property.

A dependence of the ordinal location of the trendlines and the strength of the convergence on the magnitude of $\psi$ is apparent. The curves are roughly organized according to the magnitude of $\psi$, with higher values of $|\psi|$ corresponding to lower curves and stronger rates of convergence. Conversely, lower values of $|\psi|$ seem to correspond to higher ordinal location and lesser rates of convergence.

The exception among the curves shown is the case where $T_P=6$ seconds (the curve that is also shown as an example in Figure 16). The case of $T_P=6$ seconds corresponds to a low magnitude of $\psi$, the sample correlation coefficient ($\Psi = -0.042$), but the power equation trendline has both a low ordinal position and a strong convergence rate.

Figure 18 provides another view of this convergence process illustrated by the increasing concentration of the conditional probability density distributions about the principal axes as $\sigma$ increases. Furthermore, the conditional probability about the minor axes trends toward zero and widens with increasing $\sigma$.

The curves in Figure 18 are developed from Rayleigh statistics applied to variances along axes oriented at angle $\Theta$ in equation 18. All values are normalized by the values aligned along the principal axes. The role of $m_{xx}$, $m_{yy}$ and $\Re(m_{xy})$ in equation 18, leading to Figure 18, reveals that both $\phi$ and $\psi$ contribute to the character (ordinal location and strength) of the convergence process.

The behavior of the convergence process in terms of $\phi$ and $\psi$ is not further explored in this paper. It is the opinion of this author that the observation of overarching significance is that there is a convergence process and that it appears to be a general property. The importance of this convergence property is that it increases our confidence in our joint probability findings just where they are of greatest importance, as they assume extreme values.
The role of the degrees-of-freedom in the joint process convergence is best illustrated by a finite term series approximation to equations 25 and 26. The conditions that lead to a maximum of some joint process (e.g., the Pythagorean process) impose restrictions on the effective phase angles of the Fourier terms. Correspondence of the Fourier-Stieltjes sum to an extreme value means that many (perhaps most) of the terms are constrained within a narrow range of phase angles rather than being random phase variables over the interval between 0 and 2π. These constraints may be viewed as a reduction in the degrees of freedom of the Fourier-Stieltjes process. The more extreme the joint outcome, the greater are the constraints on the phase angles and the greater is the reduction in the effective degrees of freedom.

The interested reader may also wish to consult the discussion of phase co-factors in Hutchison (2002) and also the discussion of the equivalent irregular wave in that same reference.

Figure 18: Example where T_x=7 sec, T_y=6 sec and T_P=11 sec showing increasing concentration of conditional probability density as the σ-levels increase
PROBABILITY DISTRIBUTIONS

A major goal of establishing principal axes for our seakeeping process of interest is to determine the principal process variance from the underlying basis processes. Useful statistics of the principal process can then be obtained from the Rayleigh probability distribution.

Equation 18 provides the variance for a 2-D combined process along an axis oriented at angle $\Theta$ with the x-axis. Equation 18 is valid regardless of whether $\Theta$ corresponds to a principal axis or not. If $\Theta=0^\circ$ or $\Theta=180^\circ$, then $m_{\sigma\sigma} = m_{xx}$. If $\Theta=90^\circ$ or $\Theta=270^\circ$, then $m_{\sigma\sigma} = m_{yy}$. Similar observations apply to the 3-D case described by equation 19. What is of importance is that if the basis processes are second-order stationary Gaussian processes, then the combined process amplitudes are Rayleigh distributed.

As a demonstration of the Rayleigh character of processes realized along axes at arbitrary orientations, Figure 19 shows the sample cumulative probability distributions for axes oriented at ten degree intervals from $0^\circ$ to $170^\circ$ for an example case where $T_x=7$ seconds, $T_y=6$ seconds and $T_P=11$ seconds. Also shown in Figure 19 are the residuals of the sample cumulative distributions with respect to the theoretical Rayleigh cumulative distribution. The sample cumulative distributions are tightly organized about the theoretical Rayleigh distribution, with extreme residuals ranging from -0.028 to +0.046. Furthermore, as sigma increases above a value of $\sigma \approx 2.5$, a distinct trend towards reduced residuals develops, with the magnitude of residuals at $\sigma \approx 3.5$ being 0.0012 or less.

The Rayleigh distribution of the process along the principal axis is illustrated for a variety of $T_x$, $T_y$ and $T_P$ parameter cases in Figure 20(a-d). Each panel of Figure 20 shows comparisons between the sampled and Rayleigh theoretical cumulative distributions, the Rayleigh theoretical probability density distribution, and the residuals of the sampled cumulative distribution compared to the Rayleigh theoretical distribution. In all cases the correlation between the sampled and theoretical distributions are high (in excess of 0.9995) and the residuals trend towards zero with increasing sigma above $\sigma=2.0$. Altogether these figures convincingly support the Rayleigh character of the processes along the principal axis.
Figure 20 shows an example distribution of Rayleigh statistics as a function of the angle of the axis in the x-y plane. The statistics are based on process variances obtained using equation 19 and normalized by the statistics corresponding to the principal axes.

Figure 22 shows the same distributions shown in Figure 21 but in polar coordinates. In Figure 22 the radius to any cumulative probability level measures the non-dimensional response, $\sigma$, of interest at the given angle. A graph such as Figure 21 is useful for actual engineering work but the 'peanut' diagram in Figure 22 provides a powerful mental image that may be helpful in remembering and understanding the character of the joint statistics of Pythagorean processes.
CONCLUSION

For joint, orthogonal, second-order stationary, Gaussian processes, a procedure has been described for determining, from the cross co-spectral moments of orthogonal basis processes, the principal axis for the Pythagorean combined process. Once the principal axis has been determined, it is possible to determine the variance (and higher order moments) of the process along the principal axis. Rayleigh statistics of this principal process may then be determined on an equivalent basis as those along the basis process axes.

Using samples obtained from Fourier-Stiljes simulations and conditional probability distributions
based on cross co-spectral moments, the convergence property for joint processes, and in particular the convergence onto the principal axis for extreme values of the Pythagorean process, has been demonstrated.

Finally, the Rayleigh character of the response processes along all axes, including the principal axes, was demonstrated.

The method described herein extends the method of cross co-spectral moments to the class of Pythagorean processes that includes motions, velocities, accelerations and jerks for both translations and rotations. However, the method of initially undetermined coefficients that are then determined by ascertaining the physical conditions resulting in maximization of the variance of the combination process, most likely has additional useful applications not yet discovered. Candidate processes of considerable interest, which are recommended for future research, include principal stresses and Von Mises stresses. Pitoiset et al. (1998) have presented an essentially identical method for Von Mises stresses and rainflow fatigue.

ACKNOWLEDGMENTS

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REFERENCES


EXAMPLES

A research vessel is operating at a 45° heading (bow quartering seas) at zero speed (station keeping). The sea condition is characterized by a long-crested Bretschneider spectrum with a 7 second peak period. The scientists are interested in establishing the orientation of the principal axis in the horizontal plane for rotations in order to orient some inertial instrumentation for minimum noise. The cross spectral moments for roll and pitch are:

Roll and Pitch Cross Spectral Moments, (deg)², for TP = 7 seconds at 45° Heading

<table>
<thead>
<tr>
<th>Roll</th>
<th>Pitch</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.976 + i 0.000</td>
<td>2.636 + i 0.656</td>
</tr>
<tr>
<td>2.636 - i 0.656</td>
<td>2.223 + i 0.000</td>
</tr>
</tbody>
</table>

This results in $\phi = 17.68°$ and $\psi = 0.669$, with a principal angle of $\Theta = 43.5°$. This means that the principal axis for rotation is oriented along a line 24.0° from fore and aft. The axis with the least rotation will be oriented at right-angles to this principal axis.

For this same research vessel and operating condition there is a directional instrumentation package located over the side and forward that is sensitive to local velocity. The cross spectral moments for surge ($\hat{x}$), sway ($\hat{y}$) and heave ($\hat{z}$) velocity are:

Cross Spectral Moments, (fps)², of Surge, Sway and Heave Velocity for TP = 7 seconds at 45° Heading

<table>
<thead>
<tr>
<th>$\hat{x}$</th>
<th>$\hat{y}$</th>
<th>$\hat{z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.061 + i 0.000</td>
<td>0.066 - i 0.178</td>
<td>0.021 - i 0.057</td>
</tr>
<tr>
<td>0.066 + i 0.178</td>
<td>0.738 + i 0.000</td>
<td>0.179 - i 0.292</td>
</tr>
<tr>
<td>0.021 + i 0.057</td>
<td>0.179 + i 0.292</td>
<td>1.153 + i 0.000</td>
</tr>
</tbody>
</table>

This results in $\phi_{xy} = 85.27°$, $\phi_{xz} = 86.97°$ and $\phi_{yz} = 57.39°$, with correlation values of $\psi_{xy} = 0.313$, $\psi_{xz} = 0.078$ and $\psi_{yz} = 0.194$. The principal axis for this velocity process is oriented with an azimuth angle $\Theta_1 = 79.25°$ and an elevation of $\Theta_2 = 70.12°$. 

For this research vessel and operating condition there is a directional instrumentation package located over the side and forward that is sensitive to local velocity. The cross spectral moments for surge ($\hat{x}$), sway ($\hat{y}$) and heave ($\hat{z}$) velocity are: